WAVE PROPAGATION THROUGH A VISCOUS INCOMPRESSIBLE FLUID CONTAINED IN AN INITIALLY STRESSED ELASTIC TUBE

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ABSTRACT To have a better understanding of the flow of blood in arteries a theoretical analysis of the pressure wave propagation through a viscous incompressible fluid contained in an initially stressed tube is considered. The fluid is assumed to be Newtonian. The tube is taken to be elastic and isotropic. The analysis is restricted to tubes with thin walls and to waves whose wavelengths are very large compared with the radius of the tube. It is further assumed that the amplitude of the pressure disturbance is sufficiently small so that nonlinear terms of the inertia of the fluid are negligible compared with linear ones. Both circumferential and longitudinal initial stresses are considered; however, their origins are not specified. Initial stresses enter equations as independent parameters. A frequency equation, which is quadratic in the square of the propagation velocity is obtained. Two out of four roots of this equation give the velocity of propagation of two distinct outgoing waves. The remaining two roots represent incoming waves corresponding to the first two waves. One of the waves propagates more slowly than the other. As the circumferential and/or longitudinal stress of the wall increases, the velocity of propagation and transmission per wavelength of the slower wave decreases. The response of the fast wave to a change in the initial stress is on the opposite direction.

INTRODUCTION

The arteries in a body are inflated with a mean pressure of approximately 100 mm Hg. This pressure creates a relatively high circumferential (hoop) stress. Also, arteries in a body are naturally under longitudinal tension. Thus, propagation of pressure waves in a viscous liquid, contained in an initially stressed (both circumferentially and longitudinally) elastic tube is a problem of interest in blood flow.

The propagation of pressure waves in an inflated elastic tube have attracted the attention of many investigators. One may find a fine historical perspective of this problem in the paper of Lambossy (1951). There is no unanimous agreement with regard to the effect of a pressure increase on the propagation velocity. However, the majority of scientists agrees that a small increment in the mean pressure decreases the velocity of propagation of waves in a tube. On the other hand, to the authors' best knowledge, there is no investigation which considers the effect of longitudinal tension on pulse propagation.

In this paper a theoretical analysis of the pressure wave propagation through a viscous liquid contained in an initially stressed tube is considered. The liquid is assumed to be Newtonian. The tube is taken to be elastic and isotropic. The analysis is restricted to tubes with thin walls and to waves whose wavelength is very large compared with the radius of the tube. It is further assumed that the amplitude of the pressure disturbance is sufficiently small so that nonlinear terms in the inertia of the fluid are negligible compared with linear ones.

In the final formulation of the theory, the initial stresses appear as independent parameters. If one assumes that the initial strains are very small, then measuring displacement components from an unstressed reference state, the values of the initial stresses can be calculated with the equations of linear elasticity. On the other hand, if the deformations leading to the initial stresses are very large [this is the situation in arteries; see for example McDonald (1960)], one cannot use linear theory to calculate the initial stresses. Then one either has to calculate initial stresses using large elastic deformations theory (see Tickner and Sacks, 1964), or has to determine them experimentally. In the forthcoming theory it is assumed that the numerical values of the initial stresses are known.

Since in the absence of the initial stresses the problem presented here reduces to the one treated by Womersley (1955 and 1957) it is expected that the results obtained here should also reduce to the corresponding ones given by Womersley. Keeping this fact in mind, in order to facilitate comparison, we tried to carry the present analysis parallel to Womersley's work as much as possible.

GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

The phenomenon which we are considering here is caused by the interactions of the fluid with its container, the tube. Therefore, a mathematical study of the problem should include statements with regard to the motion of the fluid, motion of the wall, and condition on their interface, namely, boundary conditions.

Hydrodynamics Equations. We will assume that fluid is incompressible and Newtonian. To express the problem we will use cylindrical coordinates r, θ, z. We will choose the z axis along the axis of the tube. We will consider only developed flow. Therefore, the choice of the origin is immaterial.

Fluid flow is governed by the Navier-Stokes equations and the equation of continuity. Assuming flow is axially symmetric, in the absence of the body forces, these equations read (see Goldstein, 1938):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} - \frac{u}{r^2}\right), \tag{1}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right), \tag{2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0. \tag{3}$$

Here t denotes time; u and w denote the components of the fluid velocity along r and z directions respectively; p is the pressure, ρ is the density, and ν is the kinetic viscosity of the fluid.

Equations of Motion of the Tube. As the wave travels, the tube will be deformed under the influence of the internal (elastic) and external loads. To express equilibrium of a surface element of the tube in its *deformed state* we will introduce a new coordinate system connected to the surface of the tube, see Fig. 1. We will con-

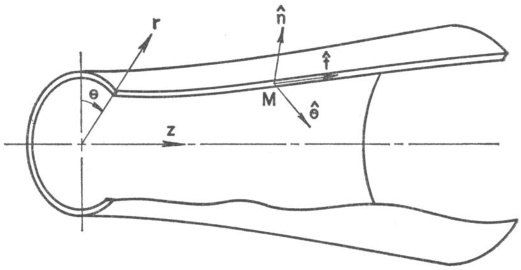

FIGURE 1 The two coordinate systems which are used for the description of the problem.

sider axially symmetric deformations. In this case, the middle surface (the surface that bisects the thickness of the tube) is obtained by rotation of a plane curve, $R = R(z, t)$, about the axis of the tube.

The position vector **M**, of any point in the middle surface is:

$$\mathbf{M} = z\hat{z} + R(z, t)\hat{r}. \tag{4}$$

Here, $\hat{z}$ and $\hat{r}$ are unit vectors of the cylinder coordinate system. Clearly, $\mathbf{M} = \mathbf{M}(z, \theta, t)$. Let us suppose that θ is kept at a constant value, while z changes. Then, equation (4) determines the meridian curve which generates the middle surface. The unit vector

$$\hat{t} = \left(\frac{\partial \mathbf{M}}{\partial z}\right)\Big/\left|\frac{\partial \mathbf{M}}{\partial z}\right| = \left(\hat{z} + \frac{\partial R}{\partial z}\hat{r}\right)\Big/\left[1 + \left(\frac{\partial R}{\partial z}\right)^2\right]^{1/2} \tag{5}$$

is tangent to the above described θ = constant curve of the middle surface. As the second base vector we choose $\hat{\theta}$, the tangential unit vector of the cylindrical coordinates. The third base vector $\hat{n}$ is taken to be the normal unit vector of the middle surface. Thus, $\hat{n}$ is orthogonal to $\hat{t}$ and $\hat{\theta}$. In terms of the vectors $\hat{r}$ and $\hat{z}$ we have:

$$\hat{n} = \left(\hat{r} - \frac{\partial R}{\partial z}\hat{z}\right)\Big/\left[1 + \left(\frac{\partial R}{\partial z}\right)^2\right]^{1/2} \tag{6}$$

To analyze internal forces, we cut from the tube an infinitesimally small element by the surface z = constant, $z + dz$ = constant, θ = constant, and $\theta + d\theta$=constant. In Fig. 2, we have shown the resultant forces and moments per unit length of the side faces of this infinitesimal element as though they were acting on the edges of the middle surface. In the following, we will assume that the thickness h of the tube is very

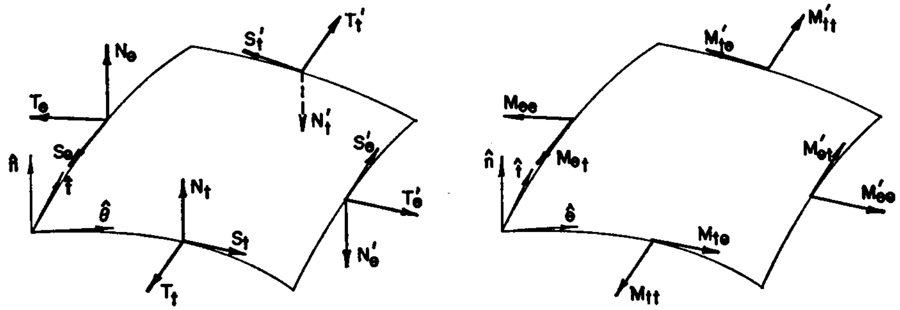

FIGURE 2 T, N, and S are the components of the resultant forces per unit length acting on the side faces of a wall elements; $M_{\theta t}$ and $M_{t\theta}$ are the bending and $M_{\theta\theta}$ and M_{tt} are the twisting moments per unit length acting on the side faces of a wall element.

small compared with the radius of the tube. Then, tensile stresses can be assumed as uniformly distributed across the thickness. Under these conditions bending moments of the stresses and shear stresses N_θ and N_t are very small and can be neglected. Thus, only three quantities T_t, T_θ, and $S_\theta = S_t$ enter into the equations of the equilibrium of the element.

From the assumed symmetry of loading and deformation we can conclude furthermore that there will be no shearing forces acting on the sides of the element. That is, we have $S_\theta = S_t = 0$. Next, we consider the external forces acting on the element. Let **P** denote the total external load per unit area of the middle surface. For axially symmetric loading, **P** will have the following form:

$$\mathbf{P} = P_t \hat{t} + P_n \hat{n}. \tag{7}$$

Summing up all of the forces in the $\hat{t}$ and $\hat{n}$ directions we obtain the following equations of equilibrium for the tube element (see Goldenveizer, 1961).

$$-\frac{\partial R}{\partial z} T_\theta + \frac{\partial}{\partial z}(RT_t) + R\left[1 + \left(\frac{\partial R}{\partial z}\right)^2\right]^{1/2} P_t = 0, \tag{8}$$

$$T_\theta / R\left[1 + \left(\frac{\partial R}{\partial z}\right)^2\right]^{1/2} - \frac{\partial^2 R}{\partial z^2} T_t \Big/ \left[1 + \left(\frac{\partial R}{\partial z}\right)^2\right]^{3/2} - P_n = 0. \tag{9}$$

The external forces acting on the tube element can be considered in two groups; inertia forces and surface forces. First let us consider inertia forces. Let $\xi = \xi(z, t)$

and $\eta = \eta(z, t)$ denote the longitudinal and radial displacements of the middle surface due to the wave motion (see Fig. 3). Then, the inertia force per unit area is

$$-\rho_0 h\left(\frac{\partial^2 \xi}{\partial t^2} + \frac{\partial^2 \eta}{\partial t^2}\right). \tag{10}$$

Here, ρ_0 denotes the density of the tube wall at the reference state. Since both density and thickness of the tube vary during the motion, strictly speaking, in the above expres-

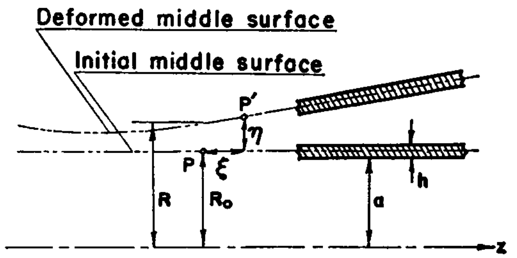

FIGURE 3 A point $P(z, R_0)$ of the middle surface of the wall at rest displaces to a position $P(z + \xi, R_0 + \eta)$ as wave travels along the tube. Note that $R(z, t)$ is not equal to $R_0 + \eta$ in general.

sion we should use their instantaneous values. However, for the motions we are considering, the changes in density and thickness are negligible. Solving equations (5) and (6) for $\hat{r}$ and $\hat{z}$, and substituting the result into expression (10), we can express inertia force in terms of $\hat{t}$ and $\hat{n}$.

$$-\frac{\rho_0 h}{\left[1 + \left(\frac{\partial R}{\partial z}\right)^2\right]^{1/2}}\left[\left(\frac{\partial^2 \xi}{\partial t^2} - \frac{\partial^2 \eta}{\partial t^2}\frac{\partial R}{\partial z}\right)\hat{t} - \left(\frac{\partial^2 \xi}{\partial t^2}\frac{\partial R}{\partial z} - \frac{\partial^2 \eta}{\partial t^2}\right)\hat{n}\right]. \tag{11}$$

The surface forces which act on the element come as the reaction of the fluid to its container. Let $\mathbf{T}_F$ denote the stress tensor of the fluid. Then, $(-\mathbf{T}_F \cdot \hat{n})$ calculated at $r = R - (h/2)$ gives us the reaction of the fluid to the inner surface of the tube element. The components of this force along $\hat{t}$ and $\hat{n}$ directions are respectively $(-\mathbf{T}_F \cdot \hat{n}) \cdot \hat{t}$ and $(-\mathbf{T}_F \cdot \hat{t}) \cdot \hat{n}$. In terms of the components of $\mathbf{T}_F$ we get:

$$(-\mathbf{T}_F \cdot \hat{n}) \cdot \hat{t} = \frac{1}{1 + \left(\frac{\partial R}{\partial z}\right)^2}\left[\frac{\partial R}{\partial z}(T_{zz} - T_{rr}) + \left(\left(\frac{\partial R}{\partial z}\right)^2 - 1\right)T_{rz}\right]_{R-(h/2)}, \tag{12}$$

$$(-\mathbf{T}_F \cdot \hat{n}) \cdot \hat{n} = \frac{1}{1 + \left(\frac{\partial R}{\partial z}\right)^2}\left[2\frac{\partial R}{\partial z}T_{rz} - T_{rr} - \left(\frac{\partial R}{\partial z}\right)^2 T_{zz}\right]_{R-(h/2)}. \tag{13}$$

Here, $[\ ]R - (h/2)$ indicates that the value of the quantity in the bracket should be calculated at $r = R - (h/2)$. The components T_{rr}, T_{zz}, and T_{rz} of $\mathbf{T}_F$, expressed in terms of velocity derivatives and pressure are given by the following relations. (Goldstein, 1938),

$$T_{rr} = -p + 2\mu \frac{\partial u}{\partial r},$$
$$T_{zz} = -p + 2\mu \frac{\partial w}{\partial z}, \qquad (14)$$
$$T_{rz} = \mu\left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right).$$

Let us substitute the values of P_t and P_n from equations (11), (12), and (13) into equations (8) and (9). Then equations of motion of the wall becomes:

$$-\frac{\partial R}{\partial z} T_\theta + \frac{\partial}{\partial z}(RT_t) - \rho_0 h R\left(\frac{\partial^2 \xi}{\partial t^2} + \frac{\partial R}{\partial z}\frac{\partial^2 \eta}{\partial t^2}\right) + \frac{R}{\left[1 + \left(\frac{\partial R}{\partial z}\right)^2\right]^{1/2}}\left[\frac{\partial R}{\partial z}(T_{zz} - T_{rr}) + \left(\left(\frac{\partial R}{\partial z}\right)^2 - 1\right)T_{rz}\right]_{R-(h/2)} = 0, \qquad (15)$$

$$\frac{T_\theta}{R\left[1 + \left(\frac{\partial R}{\partial z}\right)^2\right]^{1/2}} - \frac{\partial^2 R}{\partial z^2}\frac{T_t}{\left[1 + \left(\frac{\partial R}{\partial z}\right)^2\right]^{3/2}} - \frac{\rho_0 h}{\left[1 + \left(\frac{\partial R}{\partial z}\right)^2\right]}\left(\frac{\partial R}{\partial z}\frac{\partial^2 \xi}{\partial t^2} - \frac{\partial^2 \eta}{\partial t^2}\right) - \frac{1}{1 + \left(\frac{\partial R}{\partial z}\right)^2}\left[2\frac{\partial R}{\partial z}T_{rz} - T_{rr} - \left(\frac{\partial R}{\partial z}\right)^2 T_{zz}\right]_{R-(h/2)} = 0. \qquad (16)$$

Elasticity Relations. Stress components T_θ and T_t are related to the displacement components ξ and η with elasticity relations. In general, displacements are measured from a natural unstressed reference state for them. As the mean pressure decreases, initial tangential stress decreases. However, this is not true for the longitudinal initial stress; it can be decreased only by dissecting the arterial segments at its ends. Suppose, removing a piece of artery from a body, we obtain an unstressed reference state for it. Then, using this piece of artery, one may obtain a stress-strain relation. However, deformations required to bring back this piece of artery to its original state in the body are very large. Thus, one is forced to use the large elastic deformation theory. This theory is more difficult than the linear theory. To detour the difficulty we face here we will completely ignore the origin of the initial stresses and assume that, as the wave moves along the tube, the points of the wall undergo small deformations about the initially stressed state. Furthermore, we will assume that excess stresses due to these small deformations are related to the corresponding excess strains linearly.

Let T_{θ_0} and T_{t_0} denote circumferential and longitudinal initial stresses. Then stress-strain relations are given by the following formulas. (Goldenveizer, 1961, p. 110).

$$T_\theta - T_{\theta_0} = \frac{Eh}{1 - \sigma^2}\left(\frac{\eta}{R} + \sigma\frac{\partial\xi}{\partial z}\right) \tag{17}$$

$$T_t - T_{t_0} = \frac{Eh}{1 - \sigma^2}\left(\frac{\partial\xi}{\partial z} + \sigma\frac{\eta}{R}\right) \tag{18}$$

Here E and σ denote, respectively, Young modulus and Poisson's ratio of the wall.

Boundary Conditions. To complete the description of the problem on hand we have to supplement the equations given in previous sections by boundary conditions. Since fluid particles adhere to the inner surface of the tube, the velocity of the fluid particles on the wall must be equal to the velocity of the wall. That is,

$$u(r, z, t)|_{r=R-(h/2)} = \frac{\partial\eta}{\partial t}, \tag{19}$$

$$w(r, z, t)|_{r=R-(h/2)} = \frac{\partial\xi}{\partial t}. \tag{20}$$

As another kinematical condition we will express the fact that the component of the fluid velocity normal to the wall of the tube is equal to the normal velocity of the normal velocity of the inner surface of the tube. Since $r = R(z, t) - (h/2)$ is the inner surface of the tube, this condition can be written as (Goldstein, 1960, p. 10):

$$\frac{d}{dt}[r - R - (h/2)] = 0;$$

carrying out differentiation, we obtain

$$u - \frac{\partial R}{\partial t} + w\frac{\partial R}{\partial z} = 0. \tag{21}$$

Here u and w have to be calculated at $r = R - (h/2)$.

The conditions (19) and (20) being more general in nature, include equation (21) as a special case. However, as we will show later equation (21) yields a useful result when equations are linearized.

We will consider boundary conditions related to the coordinate z and time t after we linearize the above given equations and conditions.

LINEARIZATION OF THE EQUATIONS AND THE BOUNDARY CONDITIONS

After stating governing equations and boundary conditions we are ready to attempt to solve the problem.

However, equations of fluid motion and equations of motion of the wall are nonlinear. It is very difficult to solve them. To simplify the problem we will linearize these equations.

To start linearization, first we will expand all of the dependent variables into a power series in terms of a parameter ϵ, around a known solution of the problem. As the known solution we will take the rest state. That is, an incompressible fluid is at rest in an inflated and stretched tube. The choice of the parameter ϵ is immaterial. However, the parameter ϵ should be chosen so that for $\epsilon = 0$ all the dependent variables of the problem should give us the known solution which we started with (see Souriau, 1952).

Let us expand dependent variables of our problem into the power series of

$$\begin{aligned}
u &= u_1\epsilon + u_2\epsilon^2 + \cdots \\
w &= w_1\epsilon + w_2\epsilon^2 + \cdots \\
p &= p_0 + p_1\epsilon + p_2\epsilon^2 + \cdots \\
\xi &= \xi_1\epsilon + \xi_2\epsilon^2 + \cdots \\
\eta &= \eta_1\epsilon + \eta_2\epsilon^2 + \cdots \\
R &= R_0 + R_1\epsilon + R_2\epsilon^2 + \cdots \\
T_\theta &= T_{\theta_0} + T_{\theta_1}\epsilon + T_{\theta_2}\epsilon^2 + \cdots \\
T_t &= T_{t_0} + T_{t_1}\epsilon + T_{t_2}\epsilon^2 + \cdots
\end{aligned} \tag{22}$$

Here p_0, R_0, T_{θ_0}, and T_{t_0} are constants. Their values are given as the initial state of the system.

To carry out linearization, we will be forced to calculate the value of a certain function $f = f(r, z, t)$ at $r = R - (h/2)$. Using Taylor's theorem together with power series expansion of ϵ we find

$$f(r, z, t)|_{r=R-(h/2)} = f_0(a, z, t) + \epsilon\left[f_1(a, z, t) + R_1(z, t)\frac{\partial f_0(a, z, t)}{\partial r}\right] + 0(\epsilon^2). \tag{23}$$

This formula helps us to calculate the value of a function at a variable boundary.

Let us substitute equations (22) and (23) into the governing equations and boundary conditions given in the previous section. Collecting coefficients of the like powers of ϵ and equating them to zero we obtain sets of equations for different order of approximation.

For the zero*th* order approximation equations (1) and (2) give us:

$$\frac{\partial p_0}{\partial r} = 0, \qquad \frac{\partial p_0}{\partial z} = 0 \tag{24}$$

From equation (15) we obtain

$$\frac{T_{\theta_0}}{R_0} = p_0 = 0 \tag{25}$$

The rest of the equations do not contribute to the zero*th* order approximation.

From equations (1), (2), and (3) we get the following first order relations:

$$\frac{\partial u_1}{\partial t} = -\frac{1}{\rho}\frac{\partial p_1}{\partial r} + \nu\left(\frac{\partial^2 u_1}{\partial r^2} + \frac{1}{r}\frac{\partial u_1}{\partial r} + \frac{\partial^2 u_1}{\partial z^2} - \frac{u_1}{r^2}\right), \tag{26}$$

$$\frac{\partial w_1}{\partial t} = -\frac{1}{\rho}\frac{\partial p_1}{\partial z} + \nu\left(\frac{\partial^2 w_1}{\partial r^2} + \frac{1}{r}\frac{\partial w_1}{\partial r} + \frac{\partial^2 w_1}{\partial z^2}\right), \tag{27}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_1) + \frac{\partial w_1}{\partial z} = 0. \tag{28}$$

Similarly from equations (15) and (16) we obtain the following first order equations.

$$\rho_0 h \frac{\partial^2 \xi_1}{\partial t^2} = -\mu\left[\frac{\partial w_1}{\partial r} + \frac{\partial u_1}{\partial z}\right]_{r=a} + \frac{\partial R_1}{\partial z}\left(\frac{T_{t_0} - T_{\theta_0}}{R_0}\right) + \frac{\partial T_{t_1}}{\partial z}, \tag{29}$$

$$\rho_0 h \frac{\partial^2 \eta_1}{\partial t^2} = \left[-p_1 - 2\mu\frac{\partial u_1}{\partial r}\right]_{r=a} + \frac{R_1}{R_0} T_{\theta_0} - \frac{T_{\theta_1}}{R_0} + T_{t_0}\frac{\partial^2 R_1}{\partial z^2}. \tag{30}$$

In writing these last two equations we used equations (14). The first order stress-strain relations are:

$$T_{\theta_1} = \frac{Eh}{1-\sigma^2}\left(\frac{\eta_1}{R_0} + \sigma\frac{\partial \xi_1}{\partial z}\right) \tag{31}$$

$$T_{t_1} = \frac{Eh}{1-\sigma^2}\left(\frac{\partial \xi_1}{\partial z} + \sigma\frac{\eta_1}{R_0}\right) \tag{32}$$

The first order relations from the boundary conditions (19) and (20) are, respectively:

$$u_1|_{r=a} = \frac{\partial \eta_1}{\partial t}, \tag{33}$$

$$w_1|_{r=a} = \frac{\partial \xi_1}{\partial t}. \tag{34}$$

To the first order, equation (21) gives us:

$$u_1|_{r=a} - \frac{\partial R_1}{\partial t} = 0.$$

Comparing this result with equation (26) we see that to the first order approximation we have (see Fig. 3):

$$R_1 = \eta_1 \tag{35}$$

Equations for the second and higher order approximations may be obtained similarly. In the following, we will assume that the perturbations u_1, w_1, p_1, ξ_1, η_1, R_1, T_{θ_1}, and T_{t_1} are very small, so that we can neglect altogether high order approximations and in equation (22) we can replace ϵ with 1.

SOLUTION OF THE PROBLEM

Zeroth and First Order Solutions. The integral of the equations (24) is p_0 = constant. It is evident that the value of this constant should be equal to the initial inflation pressure of the tube. Solving equation (25) for T_{θ_0} we obtain the value of the initial circumferential stress in terms of the initial pressure and R_0.

The first order equations of the tube are coupled to the equations of motion of the fluid through the velocity derivatives they contain. Therefore, we have to start from fluid mechanics equations.

We are interested in the propagation of forced waves which are harmonic in t and z. We will assume that u_1, w_1, p_1 vary in t and z in the following manner

$$\begin{aligned} u_1 &= \bar{u}_1(r) \exp[i\omega(t - z/c)], \\ w_1 &= \bar{w}_1(r) \exp[i\omega(t - z/c)], \\ p_1 &= \bar{p}_1(r) \exp[i\omega(t - z/c)]. \end{aligned} \tag{36}$$

Here, ω denotes the circular frequency of the forced oscillation and is a real constant; c denotes the velocity of propagation of the oscillation.

The general solution of the equations (26), (27), and (28) corresponding to the forced harmonic oscillations described by equation (38) was given by Womersley (1955). This result is

$$u_1 = \left[-A \frac{\beta_0 a}{\mu\alpha_0^2} J_1\left(\beta_0 \frac{r}{a}\right) + B \frac{\beta_0}{\alpha_0 J_0(\alpha_0)} J_1\left((\alpha_0^2 + \beta_0^2)^{1/2} \frac{r}{a}\right)\right] \exp[i\omega(t - z/c)], \tag{37}$$

$$w_1 = \left[-A \frac{\beta_0 a}{\mu\alpha_0^2} J_0\left(\beta_0 \frac{r}{a}\right) + B \frac{(\alpha_0^2 + \beta_0^2)^{1/2}}{\alpha_0 J_0(\alpha_0)} J_0\left((\alpha_0^2 + \beta_0^2)^{1/2} \frac{r}{a}\right)\right] \exp[i\omega(t - z/c)], \tag{38}$$

$$p_1 = A J_0\left(\beta_0 \frac{r}{a}\right) \exp[i\omega(t - z/c)]. \tag{39}$$

Here A and B are integration constants; $J_0(x)$ and $J_1(x)$ denote the first kind zeroth and first order Bessel functions; α_0 and β_0 are dimensionless parameters and are given by the following relations.

$$\alpha_0^2 = i^3 \frac{a^2\omega}{\nu} \equiv i^3\alpha^2, \tag{40}$$

$$\beta_0 = i \frac{a\omega}{c} \equiv i\beta. \tag{41}$$

To determine the integration constants A and B we have to use the boundary conditions (33) and (34). The right-hand side of these conditions involves ξ_1 and η_1. Since the equations of motion of the wall are linear, their solutions which correspond to the forced harmonic oscillations given by the equation (36) will have the following form

$$\bar{\eta}_1 = C \exp[i\omega(t - z/c)], \tag{42}$$

$$\xi = D \exp[i\omega(t - z/c)]. \tag{43}$$

Here C and D are constants to be determined.

Substituting now equations (37), (38), (42), and (43) into the boundary conditions (33) and (34) we obtain the following two equations for the unknown constants A, B, C, and D.

$$-A\frac{\beta_0 a}{\mu\alpha_0^2}J_1(\beta_0)+B\frac{\beta_0}{\alpha_0}\frac{J_1((\alpha_0^2+\beta_0^2)^{1/2})}{J_0(\alpha_0)}-i\omega C=0, \tag{44}$$

$$-A\frac{\beta_0 a}{\mu\alpha_0^2}J_0(\beta_0)+B\frac{(\alpha_0^2+\beta_0^2)^{1/2}}{\alpha_0}\frac{J_0((\alpha_0^2+\beta_0^2)^{1/2}}{J_0(\alpha_0)}-i\omega D=0. \tag{45}$$

We will use the equations of motion of the tube wall to obtain two more equations to determine the four unknown constants. To express the excess stresses T_{θ_1} and T_{t_1} in terms of displacement derivatives we will substitute equations (31), (32), and (35) into equations (29) and (30). Carrying out this substitution, we obtain:

$$\rho_0 h\frac{\partial^2\eta_1}{\partial t^2}=\left[p_1-2\mu\frac{\partial u_1}{\partial r}\right]_a+T_{\theta_0}\frac{\eta_1}{a^2}+T_{t_0}\frac{\partial^2\eta_1}{\partial z^2}-\frac{Eh}{1-\sigma^2}\left(\frac{\sigma}{a}\frac{\partial\xi_1}{\partial z}+\frac{\eta_1}{a^2}\right), \tag{46}$$

$$\rho_0 h\frac{\partial^2\xi_1}{\partial t^2}=-\mu\left[\frac{\partial w_1}{\partial r}+\frac{\partial u_1}{\partial z}\right]_a+\frac{T_{t_0}-T_{\theta_0}}{a}\frac{\partial\eta_1}{\partial z}+\frac{Eh}{1-\sigma^2}\left(\frac{\partial^2\xi_1}{\partial z^2}+\frac{\sigma}{a}\frac{\partial\eta_1}{\partial z}\right). \tag{47}$$

In these equations we have replaced $1/R_0$ with $1/a$. This replacement is equivalent in neglecting $h/2a$ in comparison with unity.

Finally substituting equations (37), (38), (39), (42) and (43) into the equations (46) and (47), we obtain the two additional equations which are necessary to determine the constants A, B, C, and D. From equation (46) we obtain

$$\begin{aligned}A&\left\{J_0(\beta_0)+\frac{\beta_0^2}{a\alpha_0^2}[J_0(\beta_0)-J_2(\beta_0)]\right\}\\&-B\frac{\mu\beta_0}{a\alpha_0 J_0(\alpha_0)}[J_0((\alpha_0^2+\beta_0^2)^{1/2})-J_2((\alpha_0^2+\beta_0^2)^{1/2})]\\&+C\left[\left(T_{t_0}\beta_0^2+T_{\theta_0}-\frac{Eh}{1-\sigma^2}\right)\frac{1}{a^2}+\rho_0 h\omega^2\right]+D\frac{Eh}{1-\sigma^2}\frac{\sigma\beta_0}{a^2}=0.\end{aligned} \tag{48}$$

Here $J_2(x)$ denotes the first kind of second order Bessel function.

Similarly from equation (47) we find:

$$\begin{aligned}-A&\frac{2\beta_0^2}{\alpha_0^2}J_1(\beta_0)+B\frac{\mu(\alpha_0^2+2\beta_0^2)}{a\alpha_0 J_0(\alpha_0)}J_1((\alpha_0^2+\beta_0^2)^{1/2})\\&-C\frac{\beta_0}{a^2}\left(\frac{Eh}{1-\sigma^2}\sigma+T_{t_0}-T_{\theta_0}\right)+D\left(\frac{Eh}{1-\sigma^2}\frac{\beta_0^2}{a^2}+\rho_0 h\omega^2\right)=0.\end{aligned} \tag{49}$$

These last two equations together with the equations (44) and (45) constitute a system of homogeneous linear equations for the four unknown coefficients A, B, C, and D. In order to have solutions of the type prescribed by equations (36), (42), and (43) for our problem, this system of equations should have a nontrivial solution. This requires that the determinant of the coefficients of the system must be equal zero. Equating the determinant of the coefficients to zero, we obtain the *frequency equation* in which the only unknown is the velocity of propagation c. As a quick inspection of the equations (44), (45), (48), and (49) reveals that, to solve the frequency equation for c is a very difficult task. Therefore, to obtain a reasonably simple frequency equation, we will limit our investigation, hereafter, to the study of propagation of oscillations which have long wavelengths.

Long Wave Approximation. To simplify the frequency equation we will assume that the wavelengths of the oscillations are very large compared with the the radius of the tube. This requirement is equivalent to $|\beta_0| = |a\omega/c| \ll 1$. Since for blood flow, the parameter $a(\omega/\nu)^{1/2}$ is of order one, this inequality, on the other hand, implies that $a(\omega/\nu)^{1/2} \gg |a\omega/c|$. Thus, when the magnitude of $\beta_0 = i(a\omega/c)$ is very small, we can introduce the following approximations.

$$J_0(\beta_0) \simeq 1, \qquad J_1(\beta_0) \simeq \frac{\beta_0}{2}, \qquad J_2(\beta_0) \simeq \tfrac{1}{8}\beta_0^2, \qquad \alpha_0^2 + \beta_0^2 \simeq \alpha_0^2 \tag{50}$$

Carrying the relations given by equation (50) into the equations (44), (45), (48), and (49) we obtain the following set of linear equations for A, B, C, and D.

$$-\frac{\beta_0^2 a}{2\mu\alpha_0^2} A + \tfrac{1}{2}\beta_0 F_{10} B - i\omega C = 0, \tag{51}$$

$$-\frac{\beta_0 a}{2\mu\alpha_0^2} A + B - i\omega D = 0, \tag{52}$$

$$A - \frac{\mu\beta_0}{a}(2 - F_{10})B + \left(T_{t_*}\frac{\beta_0^2}{a^2} + \frac{T_{\theta_*}}{a^2} - \frac{Eh}{1-\sigma^2}\frac{1}{a^2} + \rho_0 h\omega^2\right)C + \frac{Eh}{1-\sigma^2}\frac{\sigma\beta_0}{a^2} D = 0, \tag{53}$$

$$-\frac{\beta_0^3}{\alpha_0^2} A + \frac{\mu\alpha_0^2}{2a} F_{10} B - \left(\frac{Eh}{1-\sigma^2}\sigma + T_{t_*} - T_{\theta_*}\right)\frac{\beta_0^2}{a^2} C + \left(\frac{Eh}{1-\sigma^2}\frac{\beta_0^2}{a^2} + \rho_0 h\omega^2\right)D = 0. \tag{54}$$

Here

$$F_{10} = 2J_1(\alpha_0)/\alpha_0 J_0(\alpha_0) \tag{55}$$

and Womersley (1957) contains extensive tables of this function.

Equating the determinant of the coefficients of the equations (51) through (54) to zero we obtain frequency equation for the long wave approximation.

$$\begin{vmatrix} -\frac{\beta_0 a}{\mu\alpha_0^2} & 1 & 0 & -i\omega \\ -\frac{\beta_0^2 a}{2\mu\alpha_0^2} & \frac{1}{2}\beta_0 F_{10} & -i\omega & 0 \\ 1 & -\frac{\mu\beta_0}{a}(2-F_{10}) & T_{t_0}\frac{\beta_0^2}{a^2}+\frac{T_{\theta_0}}{a^2}-\frac{Eh}{1-\sigma^2}\frac{1}{a^2}+\rho_0 h\omega^2 & \frac{Eh}{1-\sigma^2}\frac{\sigma\beta_0}{a^2} \\ -\frac{\beta_0^3}{\alpha_0^2} & \frac{\mu\alpha_0^2}{2a}F_{10} & -\left(\frac{Eh}{1-\sigma^2}\sigma+T_{t_0}-T_{\theta_0}\right) & \frac{Eh}{1-\sigma^2}\frac{\beta_0^2}{a^2}+\rho_0 h\omega^2 \end{vmatrix} = 0 \quad (56)$$

Before we attempt to expand this determinant, let us try to get some feeling about the order of magnitude of the initial stresses T_{t_0} and T_{θ_0}.

In one of the experiments which we observed on the descending aorta of a dog, a marked 4.6 cm length of aorta retracted considerably when it was dissected at both ends. When this segment was removed from the body its excised length was only 3.2 cm. This corresponds to an extension $e = 0.44$. Assuming that, even for this large deformation, the linear theory holds, taking $p_0 = 150$ mm Hg, $\sigma = 0.5$, $E = 2 \times 10^6$ dyne/cm^2, and $h/a = 0.16$, we obtain

$$\frac{T_{\theta_0}}{a} \simeq 2 \times 10^5 \text{ dyne/cm}^2 \quad \text{and} \quad \frac{T_{t_0}}{a} \simeq 3.4 \times 10^5 \text{ dyne/cm}^2$$

This rough calculation indicates that T_{t_0}/a, T_{θ_0}/a and $Eh/1 - \sigma^2\ 1/a$ have the same order of magnitude. Then, going back to the equation (56), we see that we can neglect the term $T_{t_0}\beta_0^2/a^2$ in comparison with the terms T_{θ_0}/a^2 and $Eh/1 - \sigma^2\ 1/a^2$. Again the quantity $\rho_0 h\omega^2$ which appears in the same elements of the determinant together with T_{θ_0}/a^2 and $Eh/1 - \sigma^2\ 1/a^2$ can be neglected, even for very large harmonics.

Let us substitute into the determinant the values of α_0 and β_0 in terms of α and β; (see equations (40) and (41)].

Since the determinant is equal to zero, without altering its value, we can perform on it the following operations: (*a*) multiply first column by $\mu\alpha^2/\beta$, third column by β/ω, fourth column by i/ω; (*b*) multiply second row by $1/i\beta$, third row by $\beta a/\mu\alpha^2$, fourth row by ia/μ; (*c*) replace the elements of the second row with the elements of the first row minus twice the elements of the second row; replace the elements of the third row by the elements of first row minus elements of the third row.

Furthermore noting that $1 + i\beta^2\alpha^2(2 - F_{10}) \simeq 1$, the determinant (56) becomes

$$\begin{vmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 - F_{10} & 2 & 1 \\ 0 & 1 & \left(-\frac{T_{\theta_0}}{a}+\frac{Eh}{1-\sigma^2}\frac{1}{a}\right)\frac{\beta^2}{\alpha^2\mu\omega} & 1+\frac{Eh}{1-\sigma^2}\frac{h}{a}\frac{\beta^2}{\alpha^2}\frac{1}{\mu\omega} \\ -i\beta^3 & \frac{\alpha^2}{2}F_{10} & \left(\frac{Eh}{1-\sigma^2}\frac{\sigma}{a}+\frac{T_{t_0}}{a}-\frac{T_{\theta_0}}{a}\right)\frac{\beta^2}{\mu\omega} & \left(\frac{Eh}{1-\sigma^2}\frac{\beta^2}{a^2}-\rho_0 h\omega\right)\frac{1}{\mu\omega} \end{vmatrix} = 0$$

Here we assume that the terms such as $T_{\theta 0}\beta^2/a$, $Eh\beta^2/(1 - \sigma^2)a$ are of order one.

Now let us expand this determinant by its first column. Noting that, cofactors of both of the elements 1 and $-i\beta^2$ have a magnitude of order one, we can neglect $-i\beta^2$ times its cofactor in comparison with the cofactor of 1. This gives us:

$$\begin{vmatrix} 1 - F_{10} & 2 & 1 \\ 1 & \left(-\frac{T_{\theta 0}}{a} + \frac{Eh}{1-\sigma^2}\frac{1}{a}\right)\frac{\beta^2}{\alpha^2 \mu\omega} & 1 + \frac{E\sigma}{1-\sigma^2}\frac{h}{a}\frac{\beta^2}{\alpha^2\mu\omega} \\ \frac{1}{2}F_{10} & \left(\frac{Eh}{1-\sigma^2}\frac{\sigma}{a} + \frac{T_{t0}}{a} - \frac{T_{\theta 0}}{a}\right)\frac{\beta^2}{\alpha^2\mu\omega} & \left(\frac{Eh}{1-\sigma^2}\frac{\beta^2}{a^2} - \beta h\omega^2\right)\frac{a}{\alpha^2\mu\omega} \end{vmatrix} = 0 \tag{57}$$

Expanding the determinant (57) and introducing the following dimensionless parameters

$$T_{\theta 0}\Big/\frac{Eh}{1-\sigma^2} = \tau_\theta, \qquad T_{t0}\Big/\frac{Eh}{1-\sigma^2} = \tau_t, \qquad k = \rho_0 h/\rho a. \tag{58}$$

we obtain frequency equation.

$$\frac{4(1 - F_{10})}{(1-\sigma^2)^2}[1 - \sigma^2 - \tau_\theta + \sigma(\tau_t - \tau_\theta)]\left(\frac{c_0}{c}\right)^4 + \frac{2}{1-\sigma^2}[-k(1 - F_{10})(1 - \tau_\theta) + F_{10}(2\sigma + \tau_t - \tfrac{1}{2}\tau_\theta - \tfrac{1}{2}) - 2]\left(\frac{c_0}{c}\right)^2 + F_{10} + 2k = 0. \tag{59}$$

Here

$$c_0 = \left(\frac{Eh}{2a\rho}\right)^{1/2}, \tag{60}$$

is the Moens-Korteweg formula, for the velocity of wave propagation in an incompressible, inviscid fluid enclosed in a thin walled elastic tube.

Before starting to discuss the solution of the equation (59) let us note here that for $\tau_\theta = \tau_t = 0$ this equation reduces to the frequency equation given by Womersley; in addition if we assume that the effect of wall inertia is negligible (i.e. $k = 0$) we obtain the frequency equation given by Morgan and Kiely (1954).

The roots of equation (59) are complex numbers. We will represent them as

$$\frac{c_0}{c} = X - iY$$

since

$$\exp[i\omega(t - z/c)] = \exp[i\omega(t - zX/c_0)]\exp\left[-2\pi\frac{z}{\lambda}\frac{Y}{X}\right]$$

where λ is the wavelength of the oscillation, we see that c_0/X = velocity of propagation of the wave, $\exp(-2\pi Y/X)$ = transmission per wavelength.

The equation (59) being a fourth order equation, there are four solutions for c_0/c. However, since we solve equation (59) first for $(c_0/c)^2$ and then take the square root of $(c_0/c)^2$, two of the solutions of equation (59) differ from the other two only in sign. Thus, two out of four solutions of the frequency equation give us outgoing waves, and the remaining two solutions represent incoming waves. Here we will consider only the two outgoing waves. We will denote by c_1 and c_2 their propagation velocities. The values of c_1 and c_2 will depend on the initial stresses τ_θ and τ_t, the Poisson's ratio σ, and the parameter k. For the numerical examples which we will consider here we will take $\sigma = 0.5$ and $k = 0.1$ and let τ_θ and τ_t vary.

In Fig. 4 we give the variation of c_1/c_0 versus α for an inflated tube. We see that

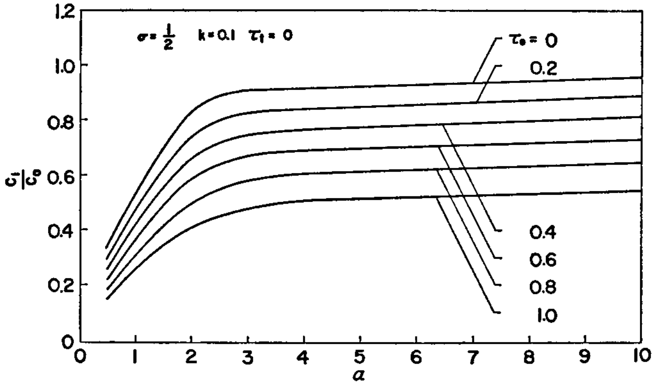

FIGURE 4 Dimensionless velocity of propagation of the first type wave, c_1/c_0, plotted against α. τ_t was taken to be zero while τ_θ took different values between zero and 1.

as the initial circumferential stress increases (i.e. as the inflation pressure increases) velocity of propagation decreases. The curve $\tau_\theta = \tau_t = 0$ coincides with the curve given Womersley (1955). The velocity of propagation is insensitive to changes in α for $\alpha > 5$.

In Fig. 5 we represent c_2/c_0 versus a for an inflated tube again. As we see from the

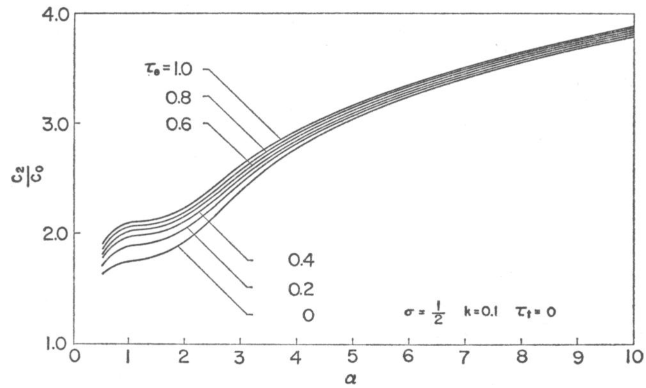

FIGURE 5 Dimensionless velocity of propagation of the second type wave, c_2/c_0 plotted against α. τ_t was taken to be zero while τ_θ took different values between zero and 1.

figure, this second wave propagates much faster than the first type of wave. Contrary to the behavior of the first type of waves, the ratio c_2/c_0 increases as τ_θ increases.

The Figs. 6 and 7 represent the transmission per wavelength versus α for the first and second type of waves respectively. The transmission of slow waves improves monotonically as α increases. On the other hand as τ_θ increases transmission decreases. The transmission behavior of the fast waves is quite different from the slow waves. For low values of α transmission is very high. As α increases transmission decreases rapidly and after passing through a minimum approximately at $\alpha = 3.5$, transmission starts to rise again steadily with α, but rather slowly. The effect of circumferential tension on the quality of transmission of the fast waves is opposite of the slow waves; as τ_θ increases transmission becomes better.

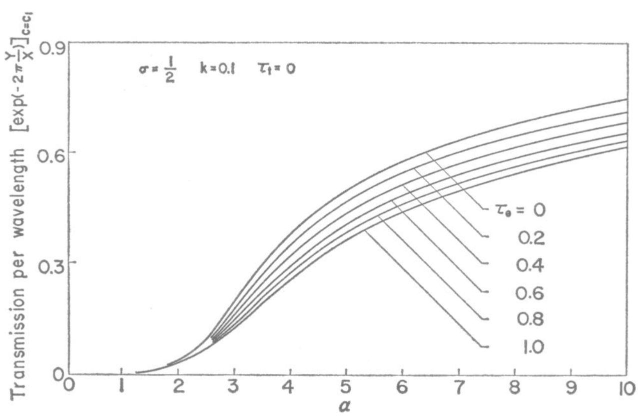

Figure 6 Transmission per wavelength plotted against α for the first type waves; $\tau_t = 0$ and τ_θ varied between zero and 1.

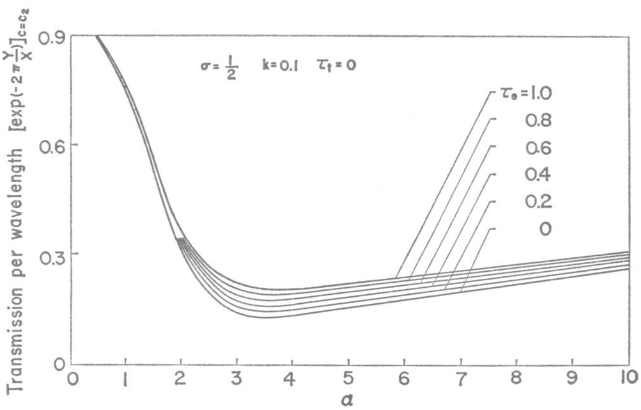

Figure 7 Transmission per wavelength plotted against α for the second type waves; $\tau_t = 0$ and τ_θ varied between zero and 1.

In Fig. 8 we show the effect of the initial longitudinal stress on c_1/c_0. Comparing this figure with the Fig. 4 we see that the effect of τ_t is the same as the effect of τ_θ. As τ_t increases the velocity of propagation c_1 decreases. On the other hand as we have shown in Fig. 9, the velocity of propagation c_2 increases with τ_t.

The effect the longitudinal initial stress τ_t, on the transmission of the slow and fast waves are represented in Figs. 10 and 11. Similar to the inflated tube case, for small α transmission is better for the fast waves; for large values of α slow waves are transmitted better. Response of the quality of transmission to an increase in τ_t for the two type of waves is similar to their response to an increase in τ_θ.

The combined effect of τ_θ and τ_t on the propagation velocities and transmissions of both type of waves are represented in Figs. 12 to 15. It is remarkable that the effects of the initial stresses are additive for both types of waves. For example, for the slow wave, the curve $\tau_\theta = \tau_t = 0.2$ of Fig. 12 is very close to the curve $\tau_\theta = 0.4$, $\tau_t = 0$

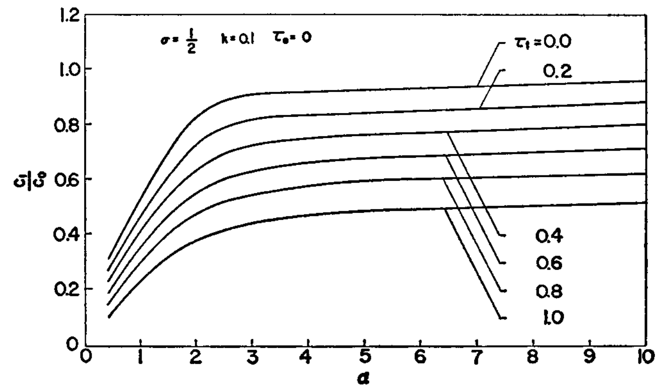

FIGURE 8 c_1/c_0 plotted against α; $\tau_\theta = 0$ and τ_t took different values between zero and 1.

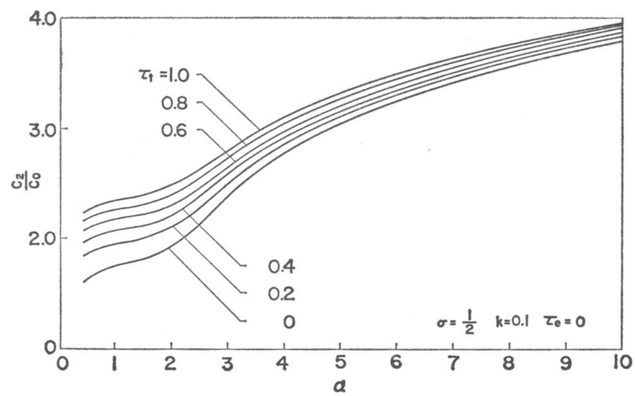

FIGURE 9 c_2/c_0 plotted against α for $\tau_\theta = 0$ and τ_t varied between zero and 1.

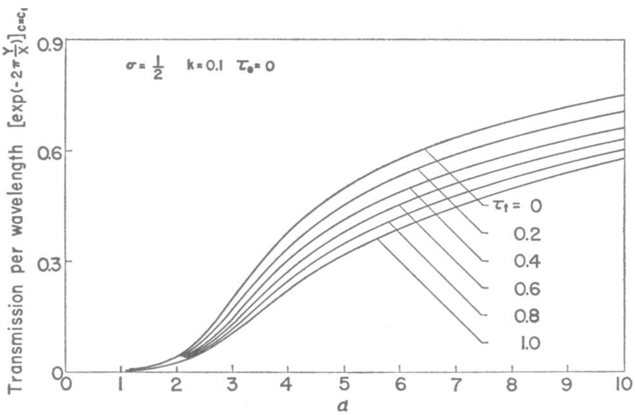

FIGURE 10 Transmission per wavelength plotted against α for the first type waves; $\tau_\theta = 0$ and τ_t varied between zero and 1.

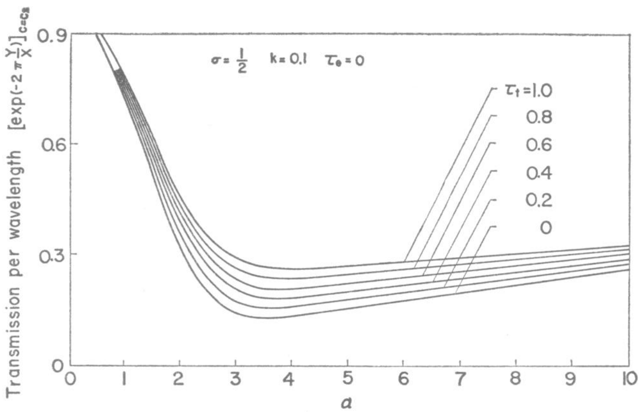

FIGURE 11 Transmission per wavelength plotted against α for the second type waves; $\tau_\theta = 0$ and τ_t varied between zero and 1.

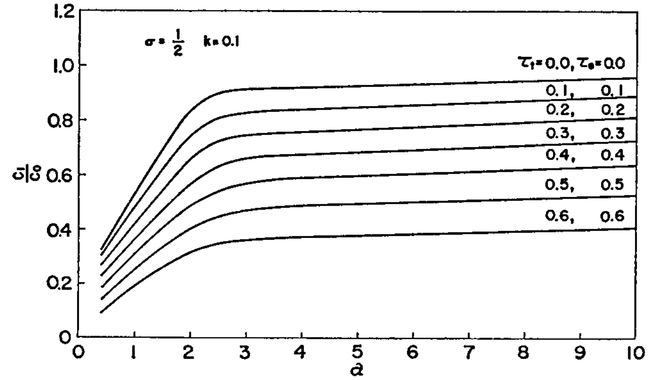

FIGURE 12 The variation of c_1/c_0 versus α under the combined effect of the longitudinal and circumferential stress.

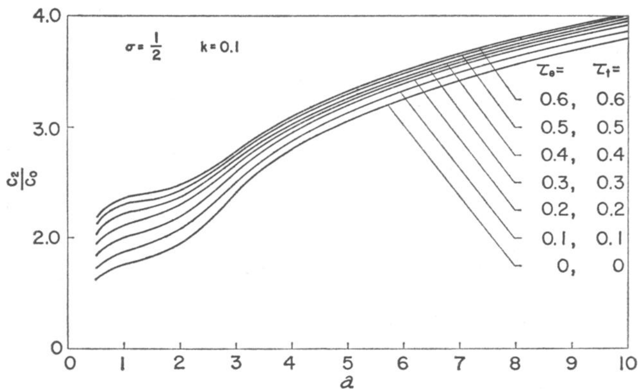

FIGURE 13 The variation of c_2/c_0 versus α under the combined effect of τ_θ and τ_t.

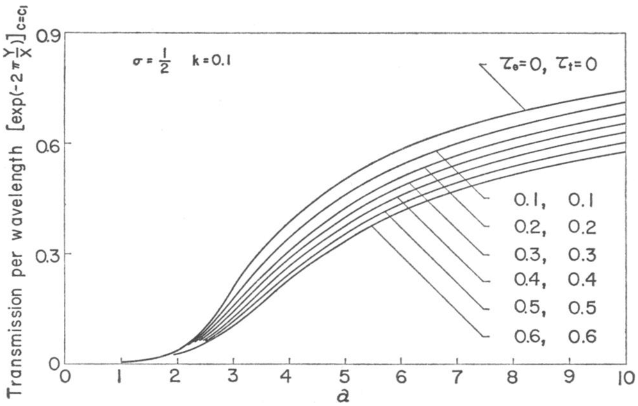

FIGURE 14 The variation of the transmission per wavelength for the first type waves under the combined effect of τ_θ and τ_t.

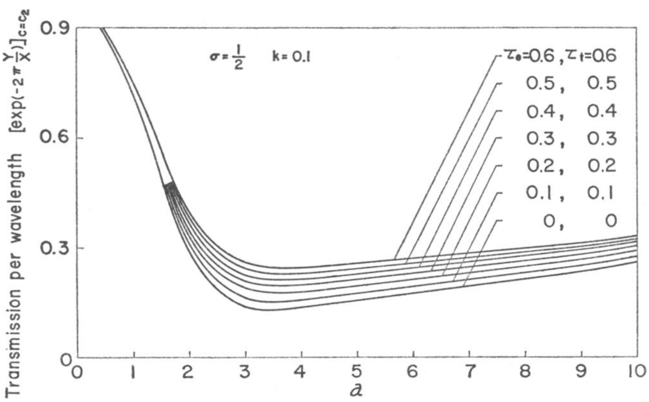

FIGURE 15 Transmission per wavelength for the second type waves under the combined effect of τ_θ and τ_t.

of Fig. 4 and also to the curve $\tau_\theta = 0$; $\tau_t = 0.4$ of Fig. 8. Similarly, for the slow waves transmission gets worse under the effect of the combined initial stresses.

It is instructive to investigate propagation properties of these two types of waves as $\alpha \to \infty$. Since $\alpha = a(\omega/\nu)^{1/2}$, we see that for constant a and ω, α goes to infinity as $\nu \to 0$. That is, $\alpha = \infty$ physically is equivalent having an inviscid fluid inside the tube. The initially stressed tube case, is rather difficult to treat without going into numerical calculations. Here, we will assume $\tau_\theta = \tau_t = 0$. Then, since $\lim_{\alpha\to\infty} F_{10} = 0$, the frequency equation (59) is simplified considerably, leading to the following solutions for $(c_0/c)^2$.

$$\left(\frac{c_0}{c}\right)^2 = \tfrac{1}{4}\{k + 2 \pm [(k + 2)^2 - 8k(1 - \sigma^2)]^{1/2}\}. \tag{61}$$

Poisson's ratio σ, can only change between zero and 1/2 and $k \geq 0$. Then, it can be shown easily that $[(k + 2)^2 - 8k(1 - \sigma^2)]^{1/2}$ is always positive and $k + 2 > [(k + 2)^2 - 8k(1 - \sigma^2)]^{1/2}$. Therefore equation (61) has always two positive roots. These two roots gives us the following values for c_1/c_0 and c_2/c_0 for outgoing waves.

$$c_1/c_0 = 2/\{(k + 2) + [(k + 2)^2 - 8k(1 - \sigma^2)]^{1/2}\}^{1/2}, \tag{62}$$

$$c_2/c_0 = 2/\{(k + 2) - [(k + 2)^2 - 8k(1 - \sigma^2)]^{1/2}\}^{1/2}. \tag{63}$$

Since the roots of equation (61) are real, we see that when fluid is inviscid both type of the waves do not attenuate. This, clearly, is in agreement with what we observe on the curves $\tau_\theta = \tau_t = 0$ of the Figs. (6), (7), (10), (11), (14), and (15) as $\alpha \to \infty$.

For $k = 0$, the equations (62) and (63) give us $c_1/c_0 = 1$ and $c_2/c_0 = \infty$ respectively. Similarly for $\sigma = 0$, (for $0 < k < 2$), from equation (62) we obtain $c_1/c_0 = 1$. Thus, the velocity c_0 can be interpreted as the velocity of propagation of the first type of waves either through an unstressed, massless tube filled with an inviscid liquid, or through an unstressed tube whose Poisson's ratio is zero and is filled with an inviscid liquid. On the other hand, for $\sigma = 0$ equation (63) gives us

$$c_2 = c_0(2/k)^{1/2} = (E/\rho)^{1/2}. \tag{64}$$

Equation (64) represents the velocity of a longitudinal wave through the tube wall. The two last results which we have obtained here for $\alpha = \infty$, $\tau_\theta = \tau_t = 0$, and $\sigma = 0$ are in agreement with the expressions given by Morgan and Ferrante (1955) as the solution of the inviscid problem.

After we determine c, we go back to the equations (51), (52), (53), and (54) and solve them for A, B, C, and D. Since this system of equations is homogeneous, we can only determine three of the unknowns in terms of the fourth one. The unknown A represents the coefficient of the pressure increment. Since the pressure is the easiest to measure among the dependent variables of our problem, here we will express B, C, and D in terms of A. From the equations (51), (52), and (53) we obtain

$$\frac{B}{A} = \frac{1}{c\rho}\frac{x[2\sigma - (1 - \tau_\theta)] + 2}{x[(1 - \tau_\theta)F_{10} - 2\sigma]}, \tag{65}$$

$$\frac{C}{A} = \frac{1}{c^2\rho}\frac{\sigma x(F_{10} - 1) + F_{10}}{x[2\sigma - (1 - \tau_\theta)F_{10}]}, \tag{66}$$

$$\frac{D}{A} = \frac{i}{c\rho\omega}\frac{2 - x(1 - \tau_\theta)(1 - F_{10})}{x[2\sigma - (1 - \tau_\theta)F_{10}]}, \tag{67}$$

where

$$x = \frac{Eh}{(1 - \sigma^2)a\rho c^2} \tag{68}$$

Let us remark here that, the initial longitudinal stress τ_t and the parameter k enter these equations implicitly through x.

To determine velocity components, first we have to carry the implications of the long wave assumption [see equation (50)] into the equations (37), (38), and (39). Doing so we find

$$u_1 = \left[-A\frac{\beta_0^2 r}{2\mu\alpha_0^2} + B\frac{\beta_0}{\alpha_0 J_0(\alpha_0)} J_1\left(\alpha_0\frac{r}{a}\right)\right] \exp\left[i\omega(t - z/c)\right], \tag{69}$$

$$w_1 = \left[-A\frac{\beta_0 a}{\mu\alpha_0^2} + B\frac{J_0\left(\alpha_0\frac{r}{a}\right)}{J_0(\alpha_0)}\right] \exp\left[i\omega(t - z/c)\right], \tag{70}$$

$$p_1 = A\exp\left[i\omega(t - z/c)\right]. \tag{71}$$

Then substituting equations (65) and (66) into equations (69) and (70) we obtain

$$u_1 = \frac{A\beta}{c\rho}i\left[\frac{r}{a} + m\frac{J_1\left(\alpha_0\frac{r}{a}\right)}{\alpha_0 J_0(\alpha_0)}\right] \exp\left[i\omega(t - z/c)\right], \tag{72}$$

$$w_1 = \frac{A}{c\rho}\left[1 + m\frac{J_0\left(\alpha_0\frac{r}{a}\right)}{J_0(\alpha_0)}\right] \exp\left[i\omega(t - z/c)\right], \tag{73}$$

where

$$m = \frac{2 + x[2\sigma - (1 - \tau_\theta)]}{x[(1 - \tau_\theta)F_{10} - 2\sigma]}. \tag{74}$$

Similarly substituting equations (66) and (67) into equations (42) and (43) we can express ξ_1 and η_1 in terms of A. The ratio ξ_1/η_1 is independent of A. Studying this ratio we may gain some insight about the nature of the first and second type of waves. From the equations (42) and (43) we see that the path of the particles of the tube wall are ellipses. The modulus of the complex number ξ_1/η_1 gives us the ratio of the major and minor axes of the ellipses. Substituting equations (66) and (67) into equations (42) and (43) and dividing resulting equations, we find that

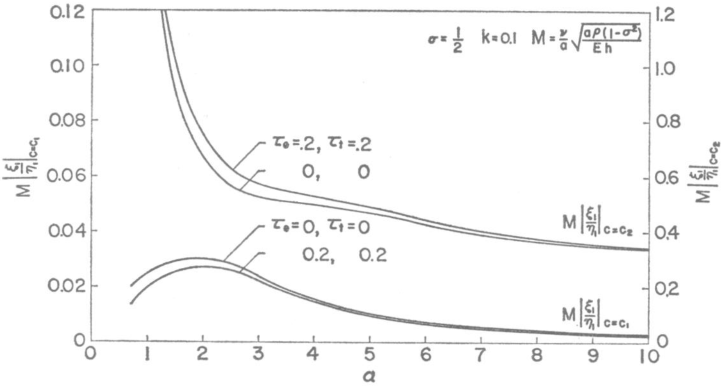

FIGURE 16 The ratio of the major axis to the minor axis of the paths of the wall particles for both type waves plotted against α for $\tau_\theta = \tau_t = 0$ and $\tau_\theta = \tau_t = 0.2$.

$$\xi_1/\eta_1 = i\frac{a}{\nu}\left[\frac{Eh}{(1-\sigma^2)a\rho}\right]^{1/2}\frac{[2 - x(1-\tau_\theta)(1-F_{10})]}{\alpha^2 x^{1/2}[\sigma x(1-F_{10}) - F_{10}]}. \tag{75}$$

In Fig. 16 we plotted $|\xi_1/\eta_1|\, \nu/a[(1-\sigma^2)a\rho/Eh]^{1/2}$ versus α for the first and second kind of waves. From these curves, we observe that the radial motion of the wall is more pronounced for the first type of wave. Since the factor $\nu/a[(1-\sigma^2)a\rho/Eh]^{1/2}$ is, in general, very small for both types of waves the paths of the particles of the wall are very flat ellipses.

CONCLUDING REMARKS

As we already remarked, for $\tau_\theta = \tau_t = 0$ equation (59) reduces to the Womersley's frequency equation. However, in neither of his publications, did Womersley mention the existence of the second type of wave.

Propagation of waves in liquid filled elastic tubes was studied experimentally by Müller (1951) and Taylor (1959). Only Müller considered the effect of the pressure of inflation on pulse propagation velocity. His experimental data agree with the findings of this theory, qualitatively for the first kind of waves. A quantitative comparison requires that the exact values of the tangential and longitudinal stresses to be known. However, in Müller's paper there is no indication about the magnitude of the longitudinal stress induced on the tube either during the assembly of the experimental setup and/or due to radial inflation.

Taylor's experiments, on the other hand, were carried out with tubes inflated to a constant internal pressure. Without knowing the exact magnitude of the longitudinal stress induced on the tube, it is not possible to compare his data with the results of this theory quantitatively. However, his measurements of the transmission per wavelength are significantly lower than what Womersley's theory predicts. Recently, similar results have been found in transmission measurements in arteries (McDonald, 1965). On the basis of data obtained by McDonald, it was concluded that the transmission in arteries was about half of what Womersley's theory predicted. Both of these experimental results are qualitatively explained by the theory presented herein.

The propagation behavior of the second type of waves is markedly different from the first type of waves. This should make it rather easy to detect them experimentally. However, there are no remarks about their existence in the above-mentioned experimental investigations. This may be attributed to the method of excitation used in these experiments. In both of these experiments oscillations are generated by applying alternating pressure to the fluid. Since the second type of waves are essentially longitudinal waves through the tube wall, modified with the existence of the fluid, they might be excited if the oscillations are generated, vibrating the tube wall longitudinally.

The longitudinal displacements predicted by the theory given here, corresponding to large pressure increments, are very large when they are compared with the observed longitudinal oscillations of arterial walls. Therefore, we join Womersley (1957) and the others in agreement on the necessity of bringing in some kind of longitudinal constraint, in order to use this theory in the study of flow of blood in arteries. Experiments carried out presently by Patel and Fry (1965) to explore the nature of the constraint on the aorta of dogs, reveal that, the simple elastic tethering model of Womersley (1957) is not an adequate one. Once the mechanism of the longitudinal tethering is clearly understood, the present theory can be extended to cover flow of the blood in arteries. However, even in that extended form, if this theory is to be useful in practice, the values of the initial stresses τ_θ and τ_t have to be determined experimentally in arterial system.

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